Deterministic and non-Deterministic Hyperuniformity in the Compact Setting

Johann S. Brauchart

j.brauchart@tugraz.at

Analysis & Number Theory TU Graz

02. March 2018

Workshop 1 – Optimal and Random Point Configurations

ICERM Semester Program on "Point Configurations in Geometry, Physics and Computer Science"

ICERM, Providence

TU Graz Graz University of Technology

FWF Der Wissenschaftsfonds.

ICERM

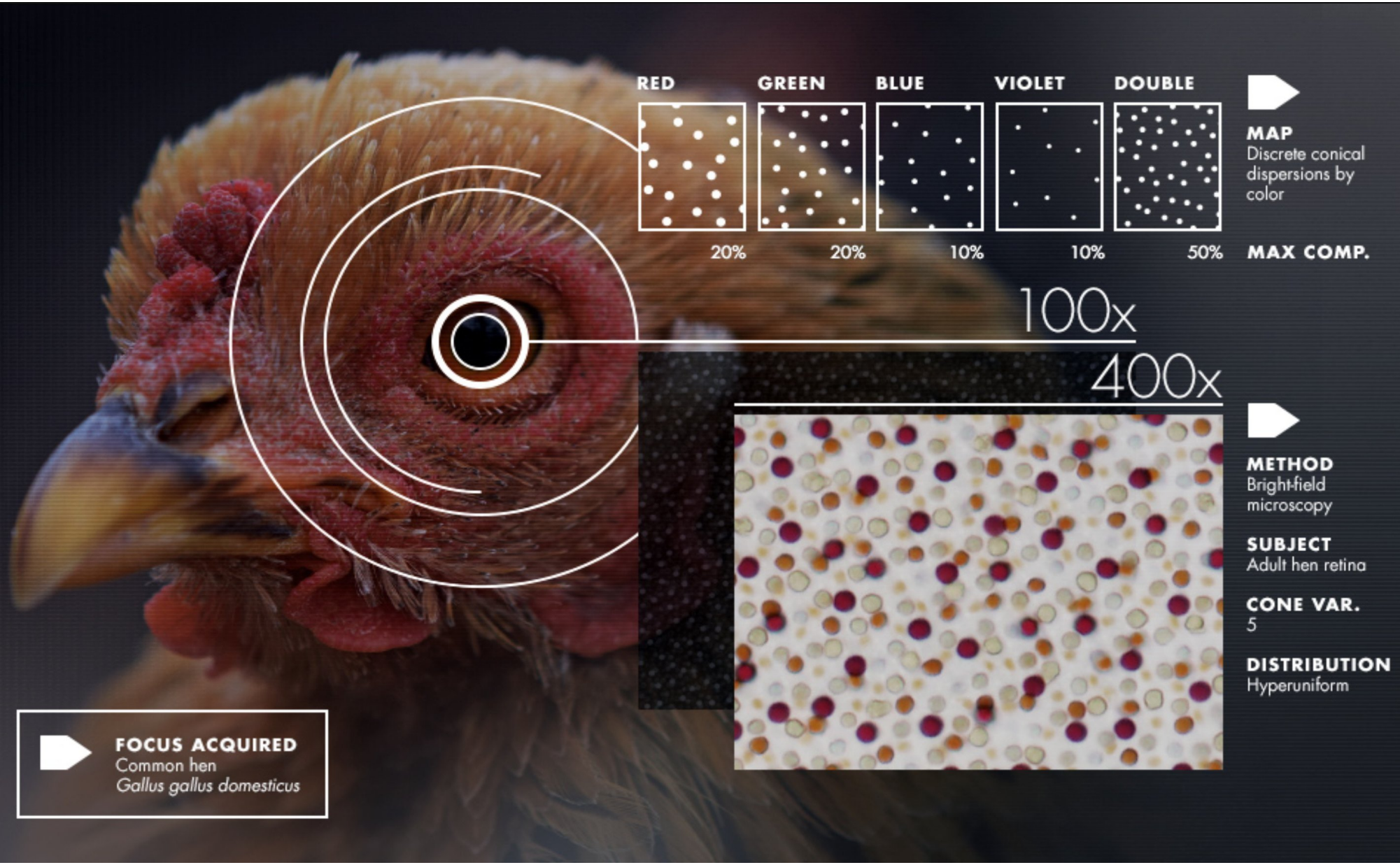
Joint work with

- Peter J. Grabner
  (Graz University of Technology)
- Wöden Kusner (Vanderbilt University)
- Jonas Ziefle (University Tübingen)

Joe Corbo, MD, PhD * stared
into the eye of a chicken
and

saw something **astonishing**.

*Professor, Pathology & Immunology at Washington University School of Medicine in St. Louis



Olena Shmahalo/Quanta Magazine; Photography: MTSOfan and Matthew Toomey

https://www.quantamagazine.org/20160712-hyperuniformity-found-in-birds-math-and-physics

PHYSICAL REVIEW E **89**, 022721 (2014)

Avian photoreceptor patterns represent a disordered hyperuniform solution to a multiscale packing problem

Yang Jiao,[1,2] Timothy Lau,[3] Haralampos Hatzikirou,[4,5] Michael Meyer-Hermann,[4]
Joseph C. Corbo,[3,*] and Salvatore Torquato[1,6,*]

[1]*Princeton Institute of the Science and Technology of Materials, Princeton University, Princeton, New Jersey 08540, USA*
[2]*Materials Science and Engineering, Arizona State University, Tempe, Arizona 85287, USA*
[3]*Department of Pathology and Immunology, Washington University School of Medicine, St. Louis, Missouri 63110, USA*
[4]*Department of Systems Immunology, Helmholtz Centre for Infection Research, Braunschweig, Germany*
[5]*Center for Advancing Electronics Dresden, TU Dresden, 01062, Dresden, Germany*
[6]*Department of Chemistry, Department of Physics, Program in Computational and Applied Mathematics, Princeton University, Princeton, New Jersey 08544, USA*

(Received 6 November 2013; published 24 February 2014)

Optimal spatial sampling of light rigorously requires that identical photoreceptors be arranged in perfectly

THE NON-COMPACT SETTING

Hyperuniformity in $\mathbb{R}^d$

Torquato and Stillinger [Physical Review E 68 (2003), no. 4, 041113]:

"A **hyperuniform** many-particle system in d-dimensional Euclidean space is one in which normalized density fluctuations are completely suppressed at very large lengths scales."

Implications

The *structure factor*

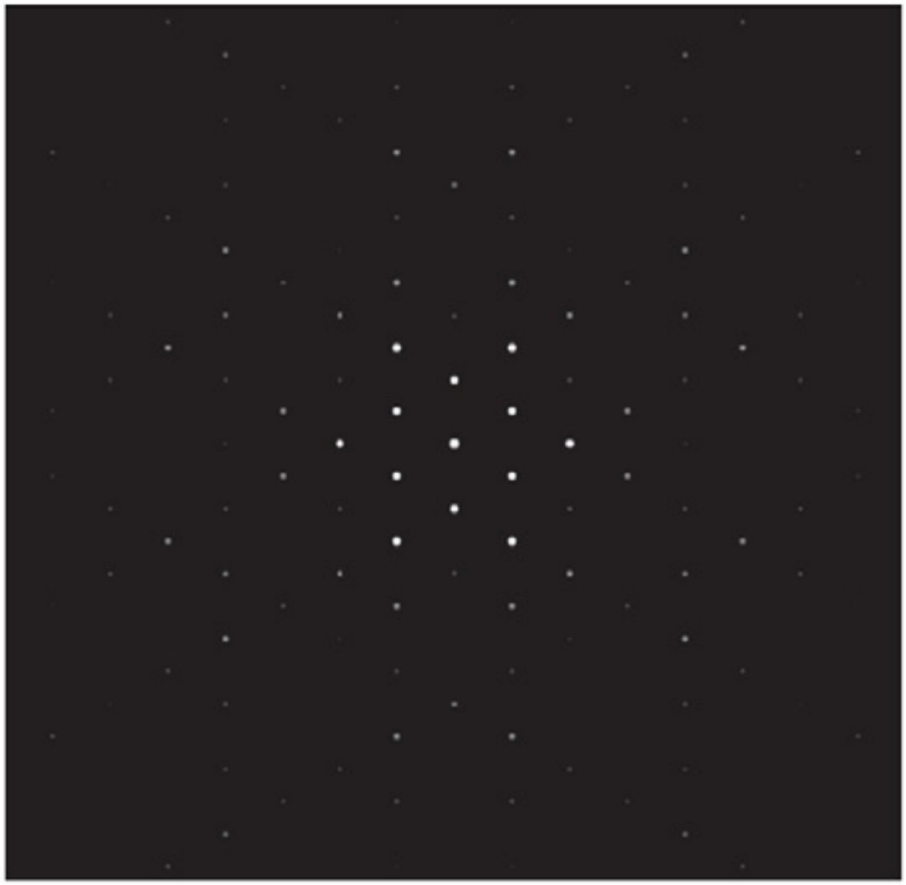
$$S(\mathbf{k}) = \lim_{B \to \mathbb{R}^d} \frac{1}{\#(B \cap X)} \sum_{\mathbf{x},\mathbf{y} \in B \cap X} e^{i\langle \mathbf{k}, \mathbf{x}-\mathbf{y} \rangle}$$

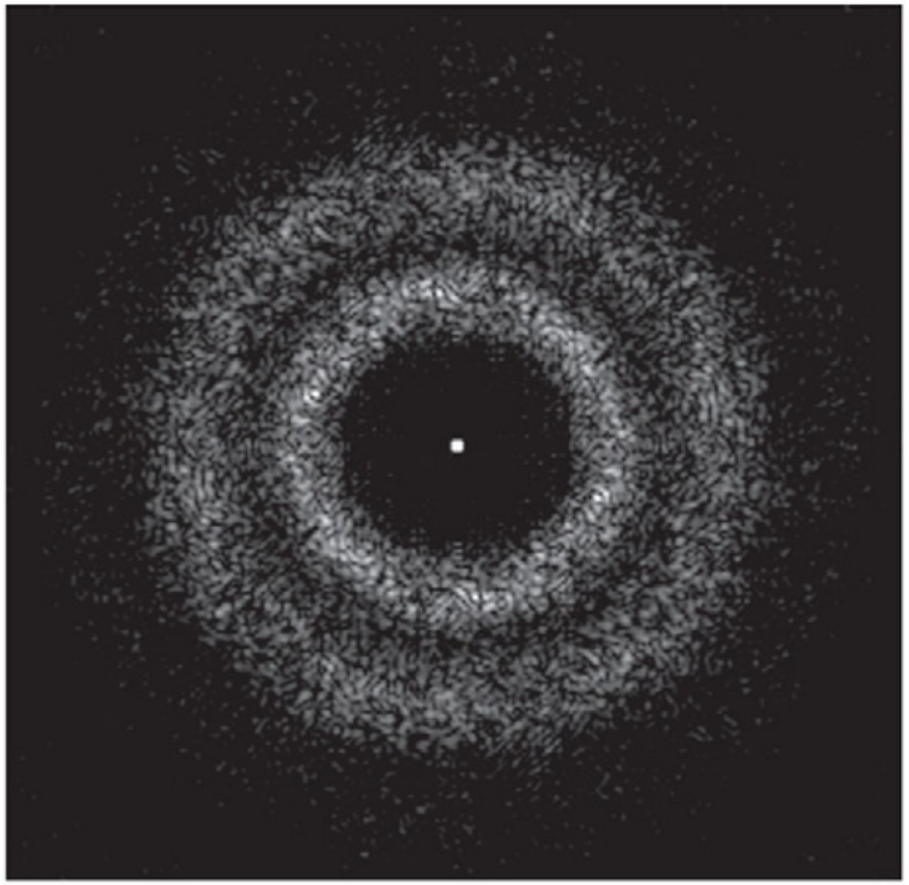
(thermodynamic limit)

tends to zero as $k \equiv |\mathbf{k}| \to 0$.

Structure Factor [a]

[a] proportional to the scattered intensity of radiation from a system of points and thus is obtainable from a scattering experiment





Scattering pattern for a **crystal** vs **disordered "stealthy" hyperuniform material**. — J. Phys.: Condens. Matter 28 (2016) 414012.

Equivalently, a

hyperuniform many-particle system

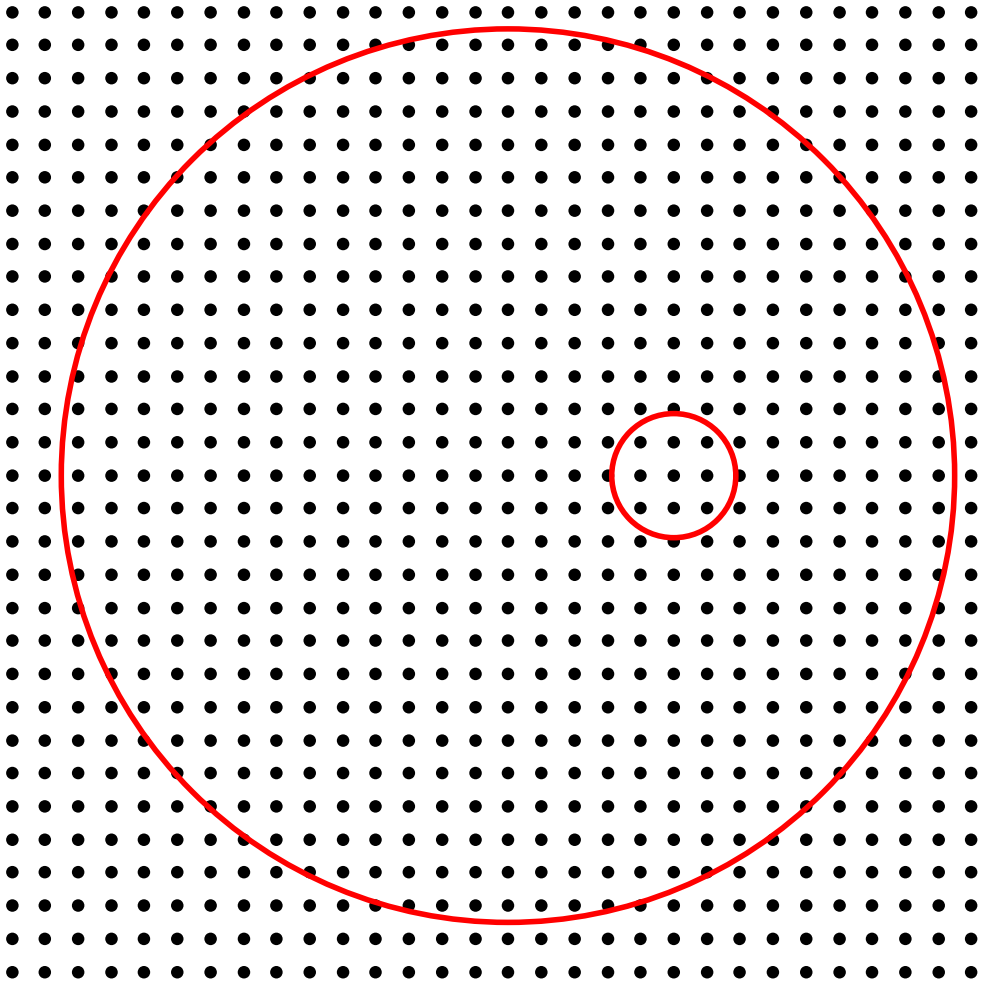
is one in which the

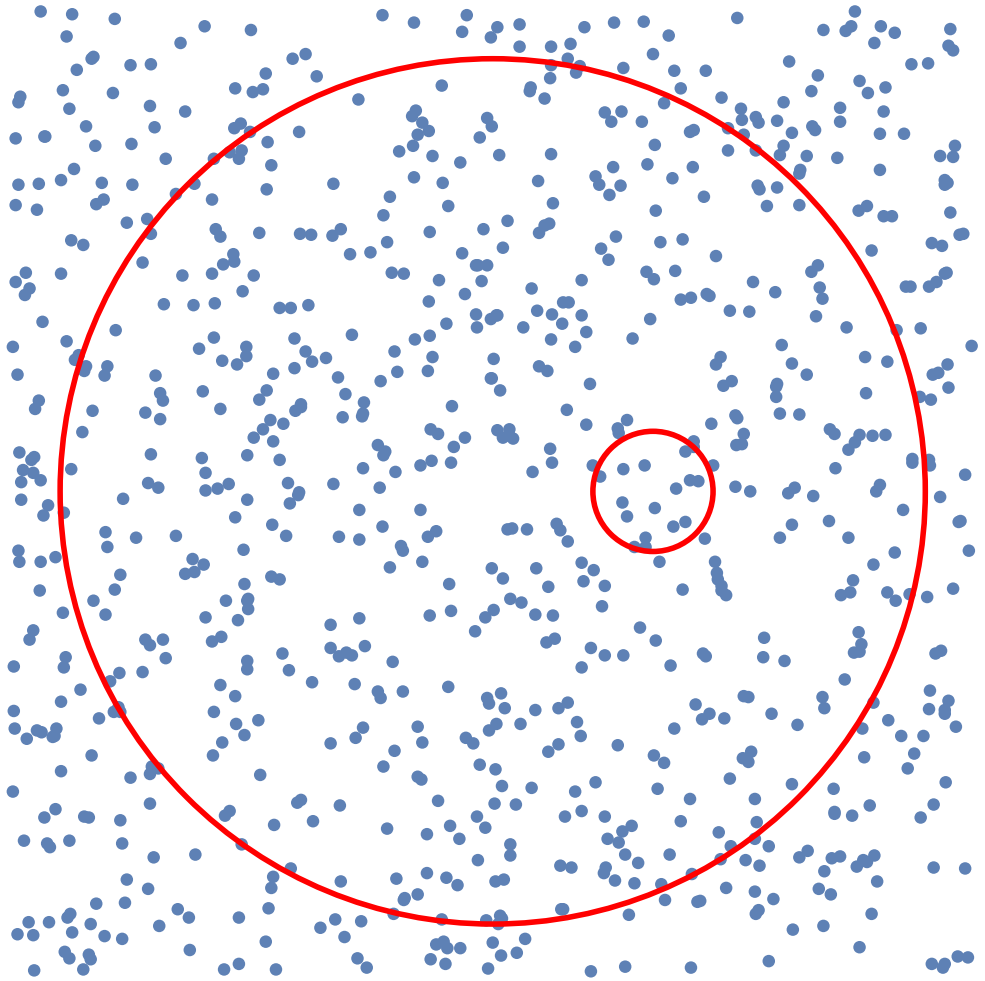
number variance $\mathrm{Var}[N_R]$ of particles

within a spherical observation window of radius R grows more slowly than the window volume in the large-R limit; i.e.,

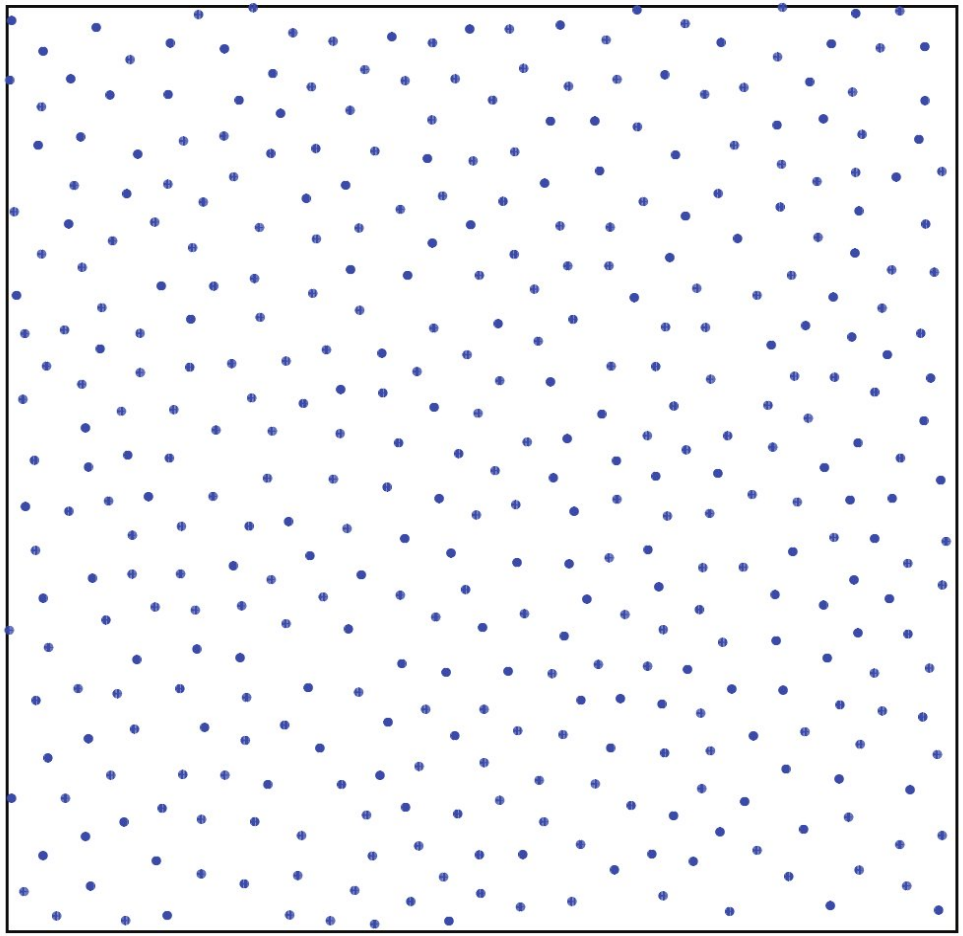
slower than R^d.

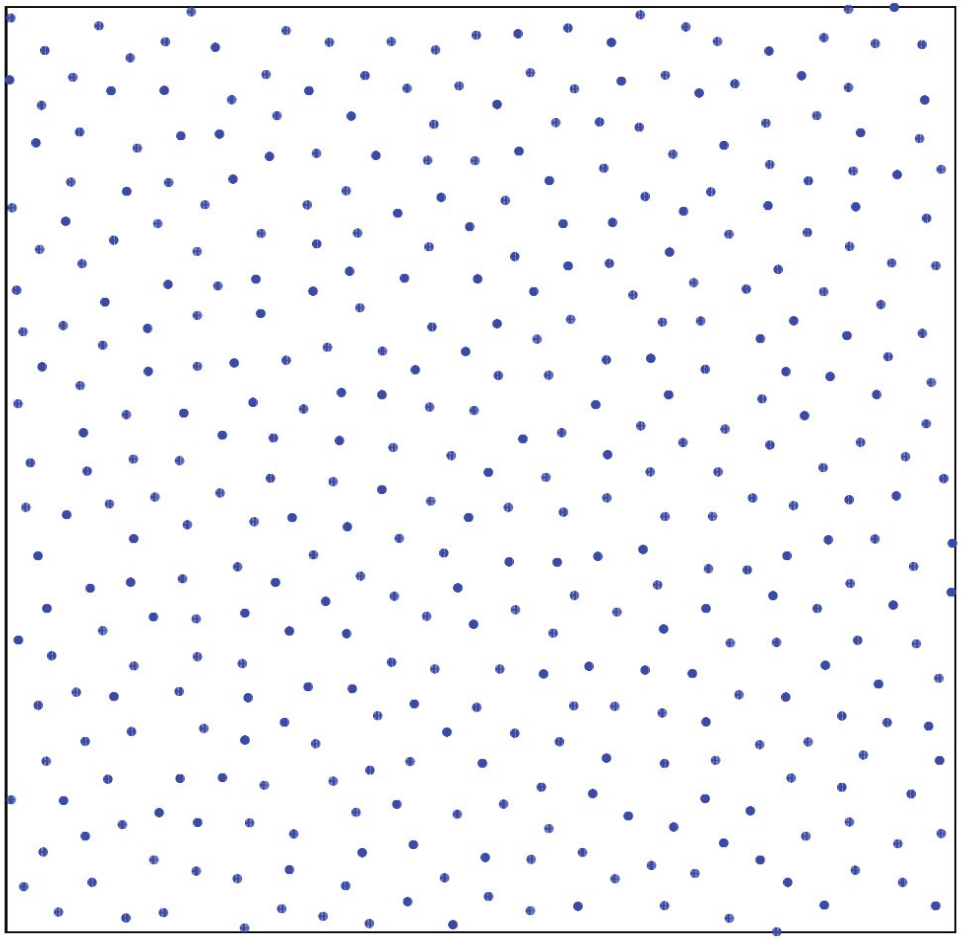
Tossing observation windows





Hyperuniformity is very subtle . . .





J. Phys.: Condens. Matter 28 (2016) 414012

Hyperuniform scaling other than surface-area growth

When

$$S(\mathbf{k}) \sim |\mathbf{k}|^{\alpha} \qquad \text{as } |\mathbf{k}| \to 0,$$

where $\alpha > 0$, the number variance has the following large-R asymptotic scaling:

$$\mathrm{Var}[N_R] \sim \begin{cases} R^{d-1} & \alpha > 1, \\ R^{d-1} \log(R) & \alpha = 1, \\ R^{d-\alpha} & 0 < \alpha < 1. \end{cases}$$

ESI Programme on "Minimal Energy Point Sets, Lattices, and Designs", 2014

THE NON-COMPACT SETTING

Optimal and random point configurations — From Statistical Physics to Approximation Theory

Institut Henri Poincaré, Paris - June 27 – July 1, 2016 – Amphi Hermite

http://djalil.chafai.net/wiki/ihp2016:start

OPTIMAL POINT CONFIGURATIONS AND ORTHOGONAL POLYNOMIALS 2017

CENTRO INTERNACIONAL DE ENCUENTROS MATEMÃTICOS (CIEM)

CASTRO URDIALES, CANTABRIA, SPAIN, APRIL 19TH-22TH, 2017

https://www.opcop2017.unican.es/

~~HYPER~~UNIFORMITY ON THE SPHERE

Infinite sequence of N-point sets on $\mathbb{S}^d$,

$$(X_N)_{N\in A}, \qquad X_N \subseteq \mathbb{S}^d, N \in A \subseteq \mathbb{N}.$$

Spherical caps

$$C(\mathbf{x}, \phi) := \left\{ \mathbf{y} \in \mathbb{S}^d \;\middle|\; \langle \mathbf{y}, \mathbf{x} \rangle > \cos(\phi) \right\}.$$

Uniform distribution on $\mathbb{S}^d$

Definition

$(X_N)_{N\in A}$ is **asymptotically uniformly distributed on** $\mathbb{S}^d$ if

$$\lim_{\substack{N\to\infty \\ N\in A}} \frac{\#\{k : \mathbf{x}_{k,N} \in B\}}{N} = \sigma_d(B)$$

for every Riemann-measurable set B in $\mathbb{S}^d$.

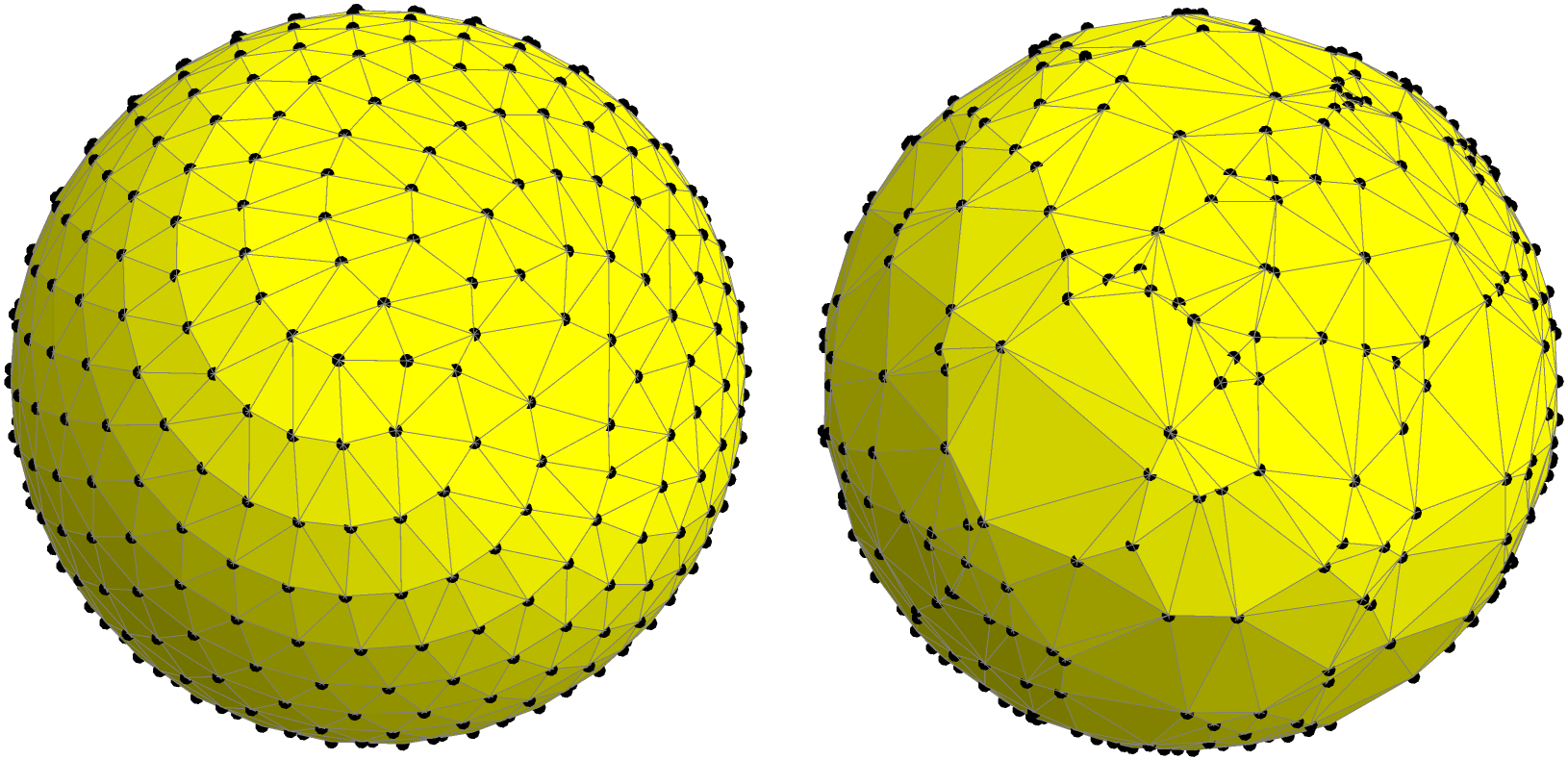
Informally: A reasonable set gets a fair share of points as N becomes large.

Equivalent definition

$(X_N)_{N\in A}$ is **asymptotically uniformly distributed on** $\mathbb{S}^d$ if

$$\lim_{\substack{N\to\infty \\ N\in A}} \frac{1}{N} \sum_{k=1}^{N} f(\mathbf{x}_k) = \int_{\mathbb{S}^d} f \, \mathrm{d}\, \sigma_d$$

for every $f \in C(\mathbb{S}^d)$.



Delaunay triangulation of spiral points (left) and i.i.d. random points (right)

PADOVA UNIVERSITY PRESS

Dolomites Research Notes on Approximation

Volume 9 · 2016 · Pages 16–49

A Comparison of Popular Point Configurations on $\mathbb{S}^2$

D. P. Hardin[a] · T. Michaels[ab] · E.B. Saff[a]

Abstract

There are many ways to generate a set of nodes on the sphere for use in a variety of problems in numerical analysis. We present a survey of quickly generated point sets on $\mathbb{S}^2$, examine their equidistribution properties, separation, covering, and mesh ratio constants and present a new point set, equal area icosahedral points, with low mesh ratio. We analyze numerically the leading order asymptotics for the Riesz and logarithmic potential energy for these configurations with total points $N < 50,000$ and present some new conjectures.

Low-discrepancy sequences on the sphere

Spherical cap $\mathbb{L}_\infty$-discrepancy

$$D^C_{\mathbb{L}_\infty}(X_N) := \sup_C \left| \frac{|Z_N \cap C|}{N} - \sigma_d(C) \right|$$

Motivated by classical (up to $\sqrt{\log N}$ optimal) results of J. Beck (1984), a sequence (X_N) is of **low-discrepancy** if

$$D^C_{\mathbb{L}_\infty}(X_N) \leq c_1 \frac{\sqrt{\log N}}{N^{1/2+1/(2d)}}.$$

Unresolved Question: Unlike in the unit cube case, there are no known explicit low-discrepancy constructions on the sphere.

Spherical cap $\mathbb{L}_\infty$ Discrepancy

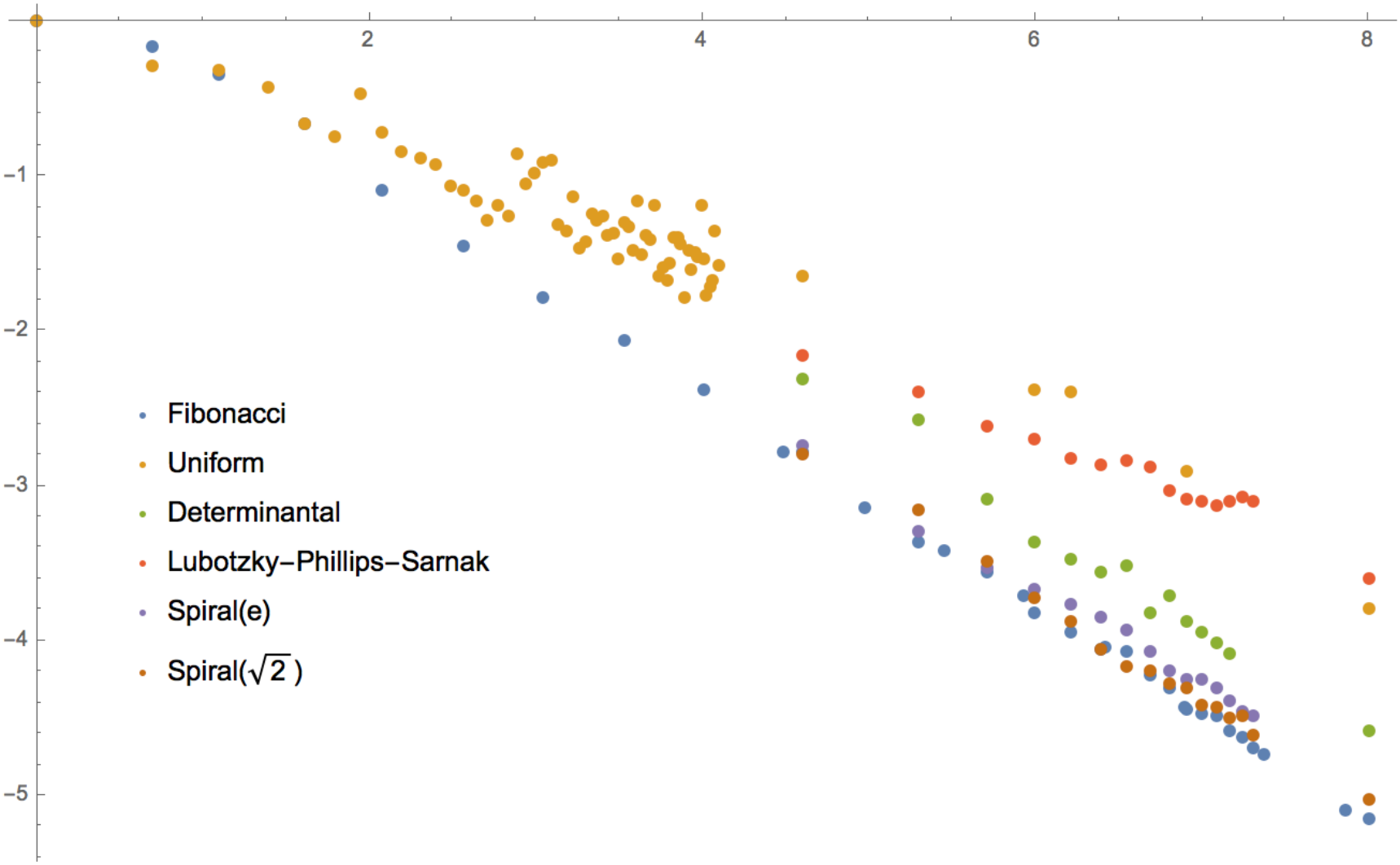
Theorem (Aistleitner-JSB-Dick, 2012)

$$D^C_{\mathbb{L}_\infty}(Z_{F_m}) \leq 44\sqrt{8} \Big/ \sqrt{F_m}$$

and numerical evidence that for some $\frac{1}{2} \leq c \leq 1$,

$$D^C_{\mathbb{L}_\infty}(Z_{F_m}) = \mathcal{O}((\log F_m)^c \, F_m^{-3/4}) \quad \text{as } F_m \to \infty.$$

RMK: A. Lubotzky, R. Phillips and P. Sarnak (1985, 1987) have $D^C_{\mathbb{L}_\infty}(X_N^{\mathrm{LPS}}) \ll (\log N)^{2/3} N^{-1/3}$ with numerical evidence indicating $\mathcal{O}(N^{-1/2})$.



ln-ln plot of spherical cap $\mathbb{L}_\infty$-discrepancy of point set families.

Should be compared with . . .

Theorem (Aistleitner-JSB-Dick, 2012)

$$\frac{c}{N^{1/2}} \leq \mathbb{E}\left[D^C_{\mathbb{L}_\infty}(X_N)\right] \leq \frac{C}{N^{1/2}}.$$

▸ Random ▸ Coulomb Surprisingly:

Theorem (Götz, 2000)

$$\frac{c}{N^{1/2}} \leq D^C_{\mathbb{L}_\infty}(X_N^*) \leq C\,\frac{\log N}{N^{1/2}},$$

X_N^* minimizing the ***Coulomb potential energy***

$$\sum_{\substack{j=1\\ j\neq k}}^{N}\sum_{k=1}^{N}\frac{1}{|\mathbf{x}_j-\mathbf{x}_k|}.$$

HYPERUNIFORMITY ON THE SPHERE

Setting

Infinite sequence of N-point sets on $\mathbb{S}^d$,

$$(X_N)_{N\in A}, \qquad X_N \subseteq \mathbb{S}^d, N \in A \subseteq \mathbb{N}.$$

Spherical caps

$$C(\mathbf{x}, \phi) := \left\{ \mathbf{y} \in \mathbb{S}^d \;\middle|\; \langle \mathbf{y}, \mathbf{x} \rangle > \cos(\phi) \right\}.$$

Asymptotic behavior of **number variance**

$$V(X_N, \phi) := \mathbb{V}_{\mathbf{x}} \left[\#\left(X_N \cap C(\mathbf{x}, \phi) \right) \right].$$

Number Variance & Uniform Distribution

$$V(X_N, \phi) = \int_{\mathbb{S}^d} \left(\sum_{n=1}^{N} \mathbb{1}_{C(\mathbf{x},\phi)}(\mathbf{x}_n) - N\,\sigma(C(\cdot, \phi)) \right) \mathrm{d}\,\sigma_d(\mathbf{x})$$

appears in classical measure of **uniform distribution**, **spherical cap $\mathbb{L}_2$-discrepancy of X_N**

$$\mathcal{D}^{C}_{\mathbb{L}_2}(X_N) := \left(\int_0^{\pi} V(X_N, \phi)\, \sin(\phi)\, \mathrm{d}\,\phi \right)^{1/2},$$

where **uniform distribution** is equivalent to

$$\lim_{N\to\infty} \frac{1}{N} \mathcal{D}^{C}_{\mathbb{L}_2}(X_N) = 0.$$

Heuristics

Heuristically, **hyperuniformity in the compact setting** should mean that the number variance $V(X_N, \phi_N)$ is of

lower order than in the i.i.d. case.

i.i.d. Case

Number Variance for i.i.d. points:

$$N\,\sigma_d(C(\cdot,\phi))\Big(1-\sigma_d(C(\cdot,\phi))\Big)$$

which has order of magnitude

- **large caps:** N
- **small caps:** $N\,\sigma_d(C(\cdot,\phi_N))$

- **threshold order:** t^d if $\phi_N = t\,N^{-1/d}$

Definition (three regimes of hyperuniformity)

$(X_N)_{N \in A}$ is

hyperuniform for large caps if

$$V(X_N, \phi) = o(N) \qquad \text{as } N \to \infty$$

for all $\phi \in (0, \frac{\pi}{2})$.

$(X_N)_{N\in A}$ is

hyperuniform for small caps if

$$V(X_N, \phi_N) = o(N\,\sigma_d(C(\cdot, \phi_N)))$$

as $N \to \infty$ for all sequences $(\phi_N)_{N\in A}$ s.t.

$$(1) \quad \lim_{N\to\infty} \phi_N = 0,$$

$$(2) \quad \lim_{N\to\infty} N \underbrace{\sigma_d(C(\cdot, \phi_N))}_{\asymp \phi_N^d} = \infty.$$

$(X_N)_{N\in A}$ is

hyperuniform for caps at threshold order[†] if

$$\limsup_{N\to\infty} V(X_N, t\,N^{-\frac{1}{d}}) = \mathcal{O}(t^{d-1})$$

as $t \to \infty$.

[†] analogous to non-compact Euclidean case

Remark

Integrating out the angular radius,

$$\int_0^{\pi} V(X_N, \phi) \sin(\phi) \, \mathrm{d}\, \phi,$$

gives back the square of the

spherical cap $\mathbb{L}_2$-discrepancy of X_N.

Stolarsky's Invariance Principle, discrepancy form

Local discrepancy function

$$D(X_N; C) := \frac{\#(X_N \cap C)}{N} - \sigma_d(C).$$

Spherical cap $\mathbb{L}_2$-discrepancy:

$$D^C_{\mathbb{L}_2}(X_N).$$

Theorem (Stolarsky,1973; JSB-Dick, 2013;)

$$\frac{1}{N^2} \sum_{j,k=1}^{N} |\mathbf{x}_j - \mathbf{x}_k| + \frac{1}{C_d} \left[D^C_{\mathbb{L}_2}(X_N) \right]^2$$

$$= \int \int |\mathbf{x} - \mathbf{y}| \, \mathrm{d}\, \sigma_d(\mathbf{x}) \, \mathrm{d}\, \sigma_d(\mathbf{y}).$$

Further properties

Laplace-Fourier series of the indicator function $\mathbb{1}_{C(\mathbf{x},\phi)}$

$$\mathbb{1}_{C(\mathbf{x},\phi)}(\mathbf{y}) = \sigma_d(C(\mathbf{x},\phi)) + \sum_{n=1}^{\infty} a_n(\phi) \underbrace{Z(d,n)\, P_n^{(d)}(\langle \mathbf{x},\mathbf{y}\rangle)}_{\frac{n+\lambda}{\lambda} C_n^{(\lambda)}(\langle \mathbf{x},\mathbf{y}\rangle),\ \lambda=\frac{d-1}{2}},$$

where for $n \geq 1$,

$$a_n(\phi) = \frac{\gamma_d}{d} \sin(\phi)^d P_{n-1}^{(d+2)}(\cos(\phi)).$$

Laplace-Fourier expansion of $V(X_N, \phi)$

$$
\begin{aligned}
&V(X_N, \phi) \\
&\quad = \int_{\mathbb{S}^d} \left(\sum_{j=1}^{N} \mathbb{1}_{C(\mathbf{x}_j, \phi)}(\mathbf{x}) - N \sigma_d(C(\cdot, \phi)) \right)^2 \mathrm{d}\, \sigma_d(\mathbf{x}) \\
&\quad = \sum_{i,j=1}^{N} \underbrace{\sum_{n=1}^{\infty} a_n(\phi)^2 \, Z(d,n) \, P_n^{(d)}(\langle \mathbf{x}_i, \mathbf{x}_j \rangle)}_{g_\phi(\langle \mathbf{x}_i, \mathbf{x}_j \rangle)},
\end{aligned}
$$

where

$$
a_n(\phi)^2 = \mathcal{O}\left(\frac{\sin(\phi)^{d-1}}{n^{d+1}} \right), \quad Z(d,n) = \mathcal{O}(n^{d-1}).
$$

Necessary condition for hyperuniformity for large caps

Theorem

If $(X_N)_{N\in\mathbb{N}}$ hyperuniform for large caps, then for every $n \geq 1$

$$s(n) := \lim_{N\to\infty} \frac{1}{N} \sum_{i,j=1}^{N} P_n^{(d)}(\langle \mathbf{x}_i, \mathbf{x}_j \rangle) = 0.$$

Proof:

$$0 = \lim_{N\to\infty} \frac{V(X_N, \phi)}{N}$$

$$\geq \underbrace{a_n(\phi)^2}_{>0} \limsup_{N\to\infty} \frac{1}{N} \sum_{i,j=1}^{N} Z(d,n)\, P_n^{(d)}(\langle \mathbf{x}_i, \mathbf{x}_j \rangle).$$

Uniform distribution of hyperuniform sequences

$(X_N)_{N\in\mathbb{N}}$ **uniformly distributed**

- iff $\dfrac{\#(X_N \cap C)}{N} \to \sigma_d(C)$
  as $N \to \infty$ for all caps C
- iff $\dfrac{1}{N^2}\sum_{i,j=1}^{N} Z(d,n)\, P_n^{(d)}(\langle \mathbf{x}_i, \mathbf{x}_j\rangle) \to 0$
  as $N \to \infty$ for all $n \geq 1$ (Weyl criterion).

Corollary

Hyperuniform $(X_N)_{N\in\mathbb{N}}$ uniformly distributed.

NOT obvious in the small caps and threshold order regimes!

Hyperuniformity of QMC design sequences

QMC design sequences for $\mathbb{H}^s(\mathbb{S}^d)$, $s > d/2$

Definition (JSB-Saff-Sloan-Womersley, 2014)

A **QMC design sequence** (X_N) **for** $\mathbb{H}^s(\mathbb{S}^d)$ satisfies

$$\sup_{\substack{f\in\mathbb{H}^s(\mathbb{S}^d),\\ \|f\|_{\mathbb{H}^s}\le 1}} \left| \frac{1}{N}\sum_{\mathbf{x}\in X_N} f(\mathbf{x}) - \int_{\mathbb{S}^d} f \,\mathrm{d}\,\sigma_d \right| \le \frac{c(s,d)}{N^{s/d}}$$

for some $c(s,d) > 0$ independent of N.

$c(s,d)$ depends on $\mathbb{H}^s(\mathbb{S}^d)$-norm.

A sequence $(Z_{N_t}^*)$ of spherical t-designs with N_t points of exactly the optimal order ($N_t \asymp t^d$) of points has the remarkable property that

$$\left| \mathrm{Q}[Z_{N_t^*}](f) - \mathrm{I}(f) \right| \leq c_s \, N_t^{-s/d} \, \|f\|_{\mathbb{H}^s}$$

for all $f \in \mathbb{H}^s(\mathbb{S}^d)$ and **all** $s > \frac{d}{2}$.

The order of N_t cannot be improved.

A sequence (X_N^*) of
maximal sum-of-distance N-point sets
define QMC rules that satisfy

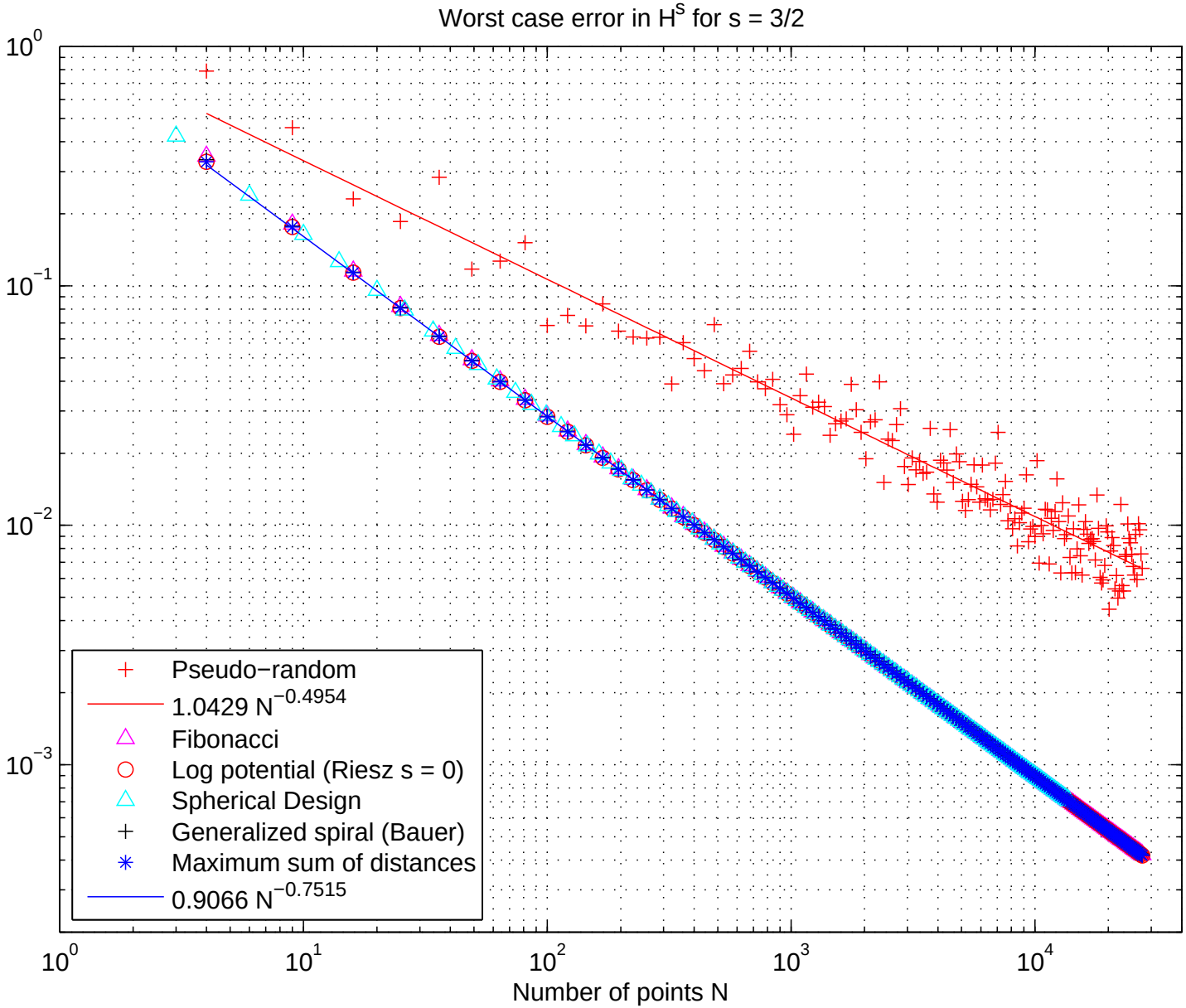
$$|\mathrm{Q}[X_{N^*}](f) - \mathrm{I}(f)| \leq c_{s'}\, N^{-s'/d}\, \|f\|_{\mathbb{H}^{s'}}$$

for all $f \in \mathbb{H}^{s'}(\mathbb{S}^d)$ and all $\frac{d}{2} < s' \leq \frac{d+1}{2}$.

The order of N cannot be improved.

Open: Determine *strength* of (X_N^*).

Numerics, I



(cf. (JSB-Saff-Sloan-Womersley, 2014))

Numerics, II: Spherical Fibonacci Points

$$4\left[D^{C}_{\mathbb{L}_2}(\mathcal{Z}_n)\right]^2 = \frac{4}{3} - \frac{1}{F_n^2}\sum_{j,k=0}^{F_n-1}\left|\boldsymbol{z}_j - \boldsymbol{z}_k\right|$$

n	F_n	$4\left[D^{C}_{\mathbb{L}_2}(\mathcal{Z}_n)\right]^2$	$F_n^{-3/2}$	$4F_n^{3/2}\left[D^{C}_{\mathbb{L}_2}(\mathcal{Z}_n)\right]^2$
3	2	6.2622e-01	3.5355e-01	1.7712
4	3	3.2188e-01	1.9245e-01	1.6725
5	5	1.2865e-01	8.9442e-02	1.4384
6	8	5.7129e-02	4.4194e-02	1.2926
7	13	2.4622e-02	2.1334e-02	1.1540
8	21	1.1107e-02	1.0391e-02	1.0688
9	34	5.0965e-03	5.0440e-03	1.0103
10	55	2.3683e-03	2.4516e-03	0.9660
11	89	1.1064e-03	1.1910e-03	0.9289
12	144	5.2192e-04	5.7870e-04	0.9018
13	233	2.4792e-04	2.8116e-04	0.8817
14	377	1.1837e-04	1.3661e-04	0.8665
15	610	5.6680e-05	6.6375e-05	0.8539
16	987	2.7240e-05	3.2249e-05	0.8446
17	1597	1.3119e-05	1.5669e-05	0.8372
18	2584	6.3331e-06	7.6130e-06	0.8318
19	4181	3.0598e-06	3.6989e-06	0.8272
20	6765	1.4808e-06	1.7972e-06	0.8239
21	10946	7.1699e-07	8.7320e-07	0.8211
22	17711	3.4756e-07	4.2426e-07	0.8192
23	28657	1.6848e-07	2.0613e-07	0.8173
24	46368	8.1756e-08	1.0015e-07	0.8162
25	75025	3.9663e-08	4.8662e-08	0.8150
26	121393	1.9257e-08	2.3643e-08	0.8145
27	196418	9.3470e-09	1.1487e-08	0.8136
28	317811	4.5399e-09	5.5814e-09	0.8133
29	514229	2.2041e-09	2.7118e-09	0.8128
30	832040	1.0708e-09	1.3176e-09	0.8127
31	1346269	5.1999e-10	6.4018e-10	0.8122
				0.7985

cf. B [Uniform Distribution Theory **6**:2 (2011)]

Sum of distances for Spherical Fibonacci points

$$\frac{1}{F_n^2}\sum_{j=0}^{F_n-1}\sum_{k=0}^{F_n-1}|\mathbf{z}_j-\mathbf{z}_k|^{2s-2} = V_{2s-2}+V_{2s-2}\sum_{\ell=1}^{\infty}\frac{(1-s)_\ell}{(1+s)_\ell}(2\ell+1)\left|\frac{1}{F_n}\sum_{k=0}^{F_n-1}P_\ell(1-\frac{2k}{F_n})\right|^2$$

$$+\,2V_{2s-2}\sum_{\ell=1}^{\infty}\frac{(1-s)_\ell}{(1+s)_\ell}(2\ell+1)\sum_{m=1}^{\ell}\frac{(\ell-m)!}{(\ell+m)!}\left|\frac{1}{F_n}\sum_{k=0}^{F_n-1}P_\ell^m(1-\frac{2k}{F_n})\,e^{2\pi i\,m\,kF_{n-1}/F_n}\right|^2.$$

On the right-hand side one has (the error of) the numerical integration rule

$$0=\int_{-1}^{1}P_\ell(x)\,\mathrm{d}\,x\approx\frac{1}{F_n}\sum_{k=0}^{F_n-1}P_\ell(1-\frac{2k}{F_n}),\qquad \ell\geq 1,$$

with equally spaced nodes in $[-1,1]$ for the Legendre polynomials $P_\ell(x)$ and the *Fibonacci lattice rule*

$$0=\int_0^1\int_0^1 P_\ell^m(1-2x)\,e^{2\pi i\,m\,y}\,\mathrm{d}\,x\,\mathrm{d}\,y\approx\frac{1}{F_n}\sum_{k=0}^{F_n-1}P_\ell^m(1-\frac{2k}{F_n})\,e^{2\pi i\,m\,kF_{n-1}/F_n}$$

based on the Fibonacci lattice points in the unit square $[0,1]^2$ for functions

$$f_\ell^m(x,y):=P_\ell^m(1-2x)\,e^{2\pi i\,m\,y},\qquad \ell\geq 1, 1\leq|m|\leq\ell.$$

Hyperuniformity of QMC design sequences

Theorem

A QMC design sequence for $\mathbb{H}^s(\mathbb{S}^d)$ with $s \geq \frac{d+1}{2}$ is hyperuniform for large caps, small caps, and caps at threshold order.

Lemma

The number variance satisfies

$$V(X_N, \phi) \ll (\sin \phi)^{d-1} N^2 \left[\mathrm{wce}(\mathrm{Q}[X_N]; \mathbb{H}^{\frac{d+1}{2}}(\mathbb{S}^d))\right]^2$$

for any N-point set $X_N \subseteq \mathbb{S}^d$ and opening angle $\phi \in (0, \frac{\pi}{2})$.

Conclusions from the WCE formula

Any QMC design sequence $(X_N)_{N\in A}$ for $\mathbb{H}^s(\mathbb{S}^d)$, $s \geq \frac{d+1}{2}$, is hyperuniform for large caps and thus

$$\lim_{\substack{N\to\infty \\ N\in A}} \frac{1}{N} \sum_{i,j=1}^{N} P_n^{(d)}(\langle \mathbf{x}_i, \mathbf{x}_j \rangle) = 0 \quad \text{for every } n \in \mathbb{N}.$$

The WCE formula yields for each fixed $n \geq 1$

$$\lim_{\substack{N\to\infty \\ N\in A}} N^{-1+\frac{s^*-(1+\varepsilon)\frac{d}{2}}{\frac{d}{2}}} \sum_{i,j=1}^{N} P_n^{(d)}(\langle \mathbf{x}_i, \mathbf{x}_j \rangle) = 0$$

for all sufficiently small $\varepsilon > 0$, where $s^* > \frac{d}{2}$ is the (finite) strength of $(X_N)_{N\in A}$.

Hyperuniform point sets on the sphere: deterministic constructions

Johann Brauchart, Peter Grabner, Wöden Kusner

(Submitted on 8 Sep 2017 (v1), last revised 12 Sep 2017 (this version, v2))

We study a generalisation of the concept of hyperuniformity to spheres of arbitrary dimension. It is shown that QMC-designs (and especially spherical designs) are hyperuniform in our sense.

Subjects:	**Classical Analysis and ODEs (math.CA)**
MSC classes:	65C05, 11K38, 65D30
Cite as:	**arXiv:1709.02613 [math.CA]**
	(or **arXiv:1709.02613v2 [math.CA]** for this version)

Submission history

From: Peter Grabner [view email]

[v1] Fri, 8 Sep 2017 09:38:38 GMT (16kb)

[v2] Tue, 12 Sep 2017 09:59:15 GMT (16kb)

Thank You!

Appendix

J. Beck, 1984

To every N-point set Z_N on $\mathbb{S}^d$ there exists a spherical cap $C \subset \mathbb{S}^d$ s.t.

$$c_1\, N^{-1/2-1/(2d)} < \left| \frac{|Z_N \cap C|}{N} - \sigma_d(C) \right|$$

and (by a probabilistic argument) there exist an N-point sets Z_N^* on $\mathbb{S}^d$ s.t.

$$\left| \frac{|Z_N^* \cap C|}{N} - \sigma_d(C) \right| < c_2\, N^{-1/2-1/(2d)} \sqrt{\log N}$$

for every spherical cap C.

Discrete Comput Geom (2012) 48:990–1024
DOI 10.1007/s00454-012-9451-3

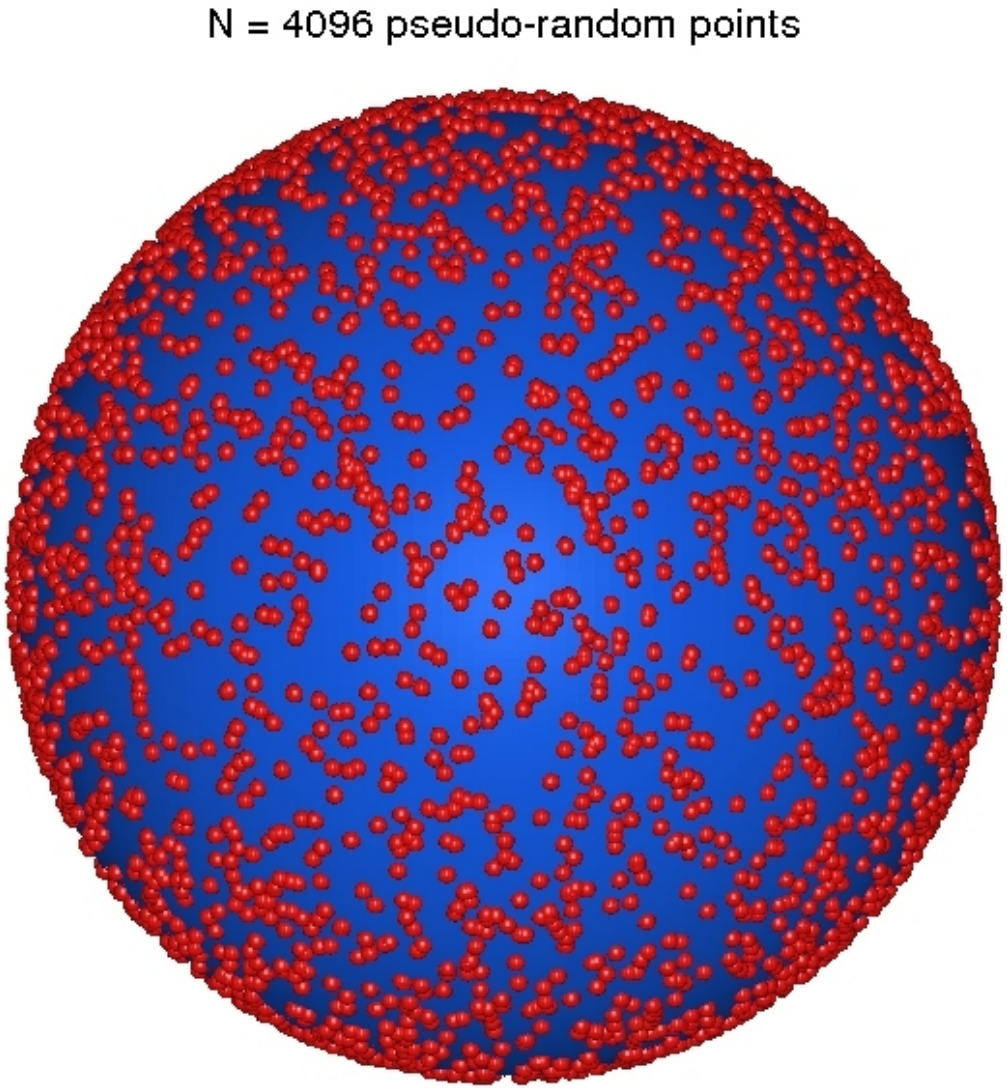
Point Sets on the Sphere $\mathbb{S}^2$ with Small Spherical Cap Discrepancy

C. Aistleitner · J.S. Brauchart · J. Dick

Received: 14 September 2011 / Revised: 31 July 2012 / Accepted: 7 August 2012 /
Published online: 28 August 2012
© Springer Science+Business Media, LLC 2012

Random Points on $\mathbb{S}^d$

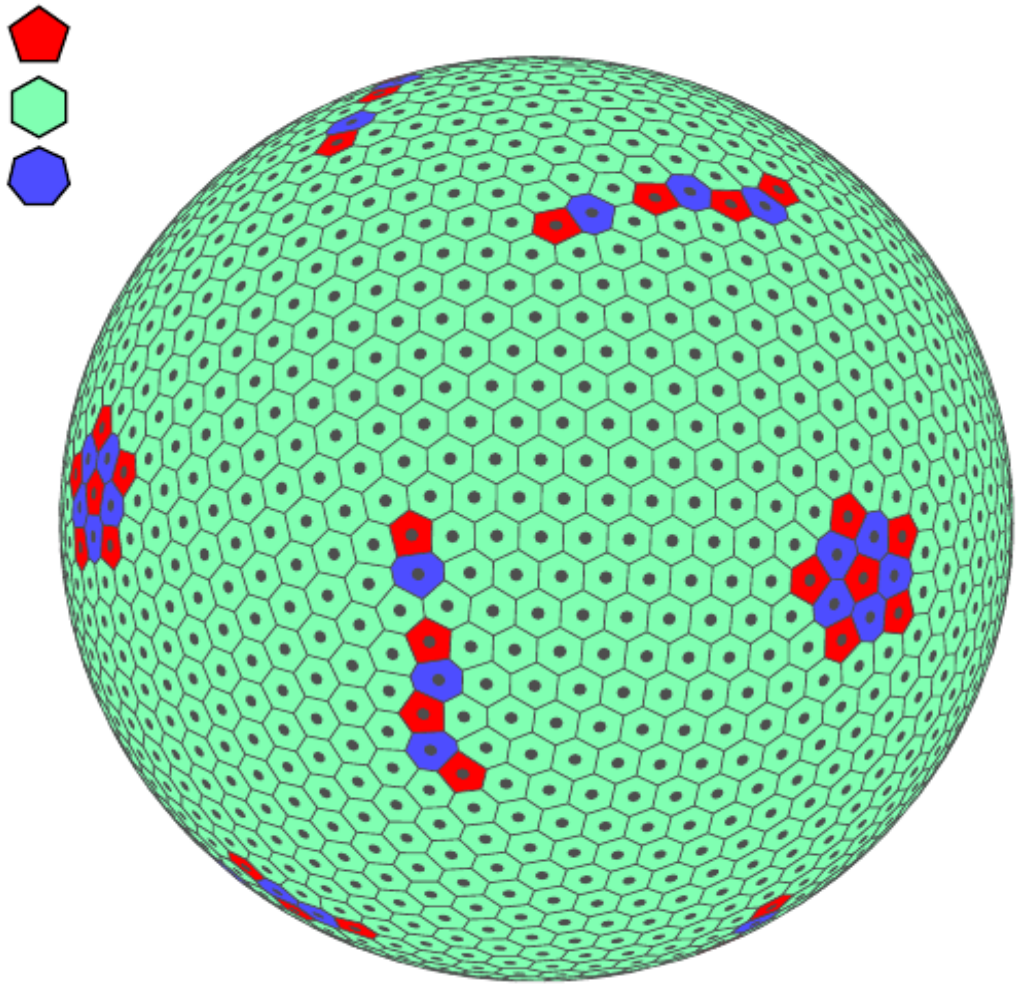
Return



$\mathbf{X}_1, \ldots, \mathbf{X}_N$ i.i.d. uniformly on $\mathbb{S}^d$

Minimum Riesz s-Energy Points on $\mathbb{S}^2$

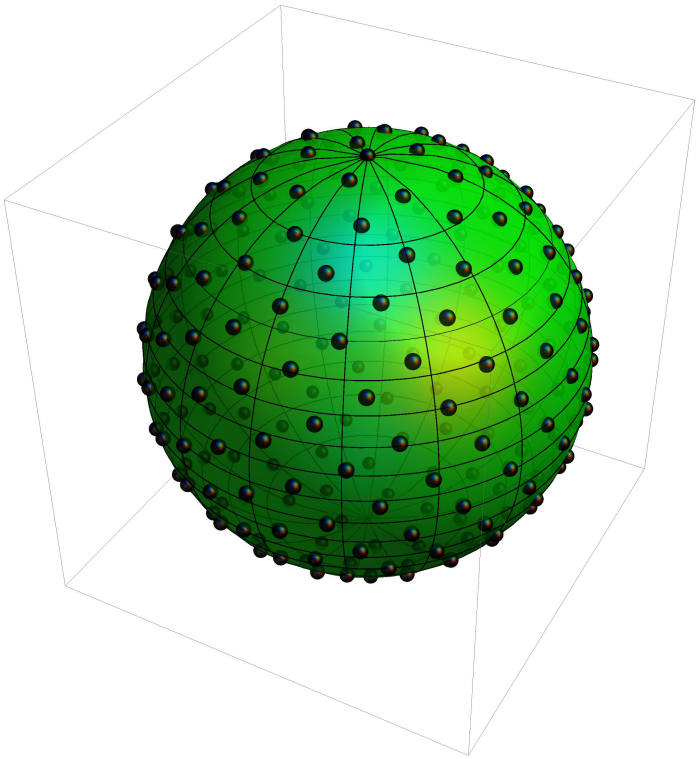
Return

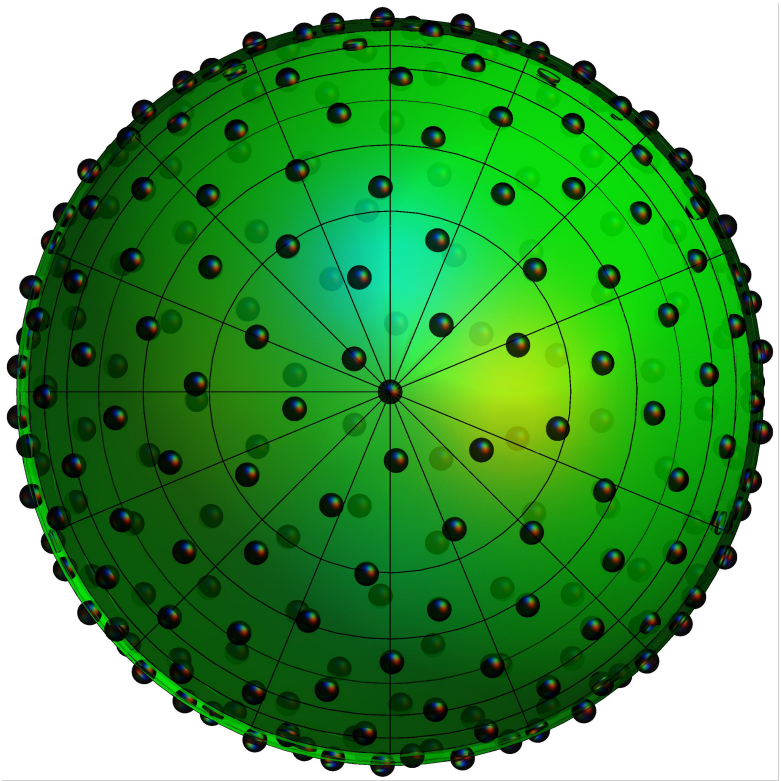


$N = 1600$, $s = 1$ (Coulomb case);
(cf. Hardin and Saff, 2004, Notices of AMS).

Spherical

Fibonacci Lattice Points





Fibonacci sequence (OEIS: A000045):

$$\begin{aligned} F_0 &:= 0, \\ F_1 &:= 1, \\ F_{n+1} &:= F_n + F_{n-1}, \qquad n \geq 1. \end{aligned}$$

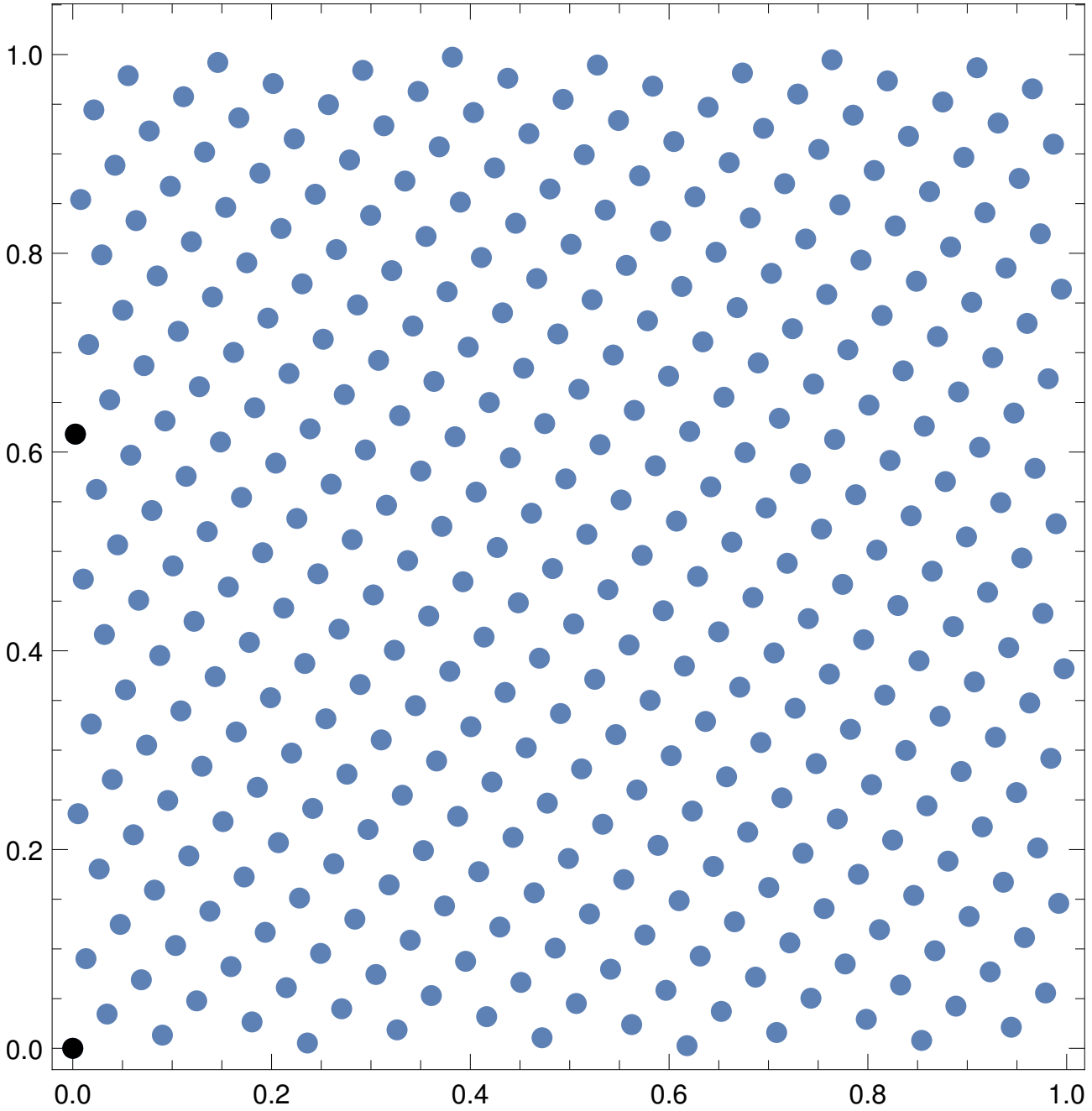
Fibonacci lattice in $[0,1]^2$

$$\mathcal{F}_n : \qquad \left(\frac{k}{F_n}, \left\{ k \frac{F_{n-1}}{F_n} \right\} \right), \quad 0 \leq k < F_n,$$

has optimal order star-discrepancy bounds:

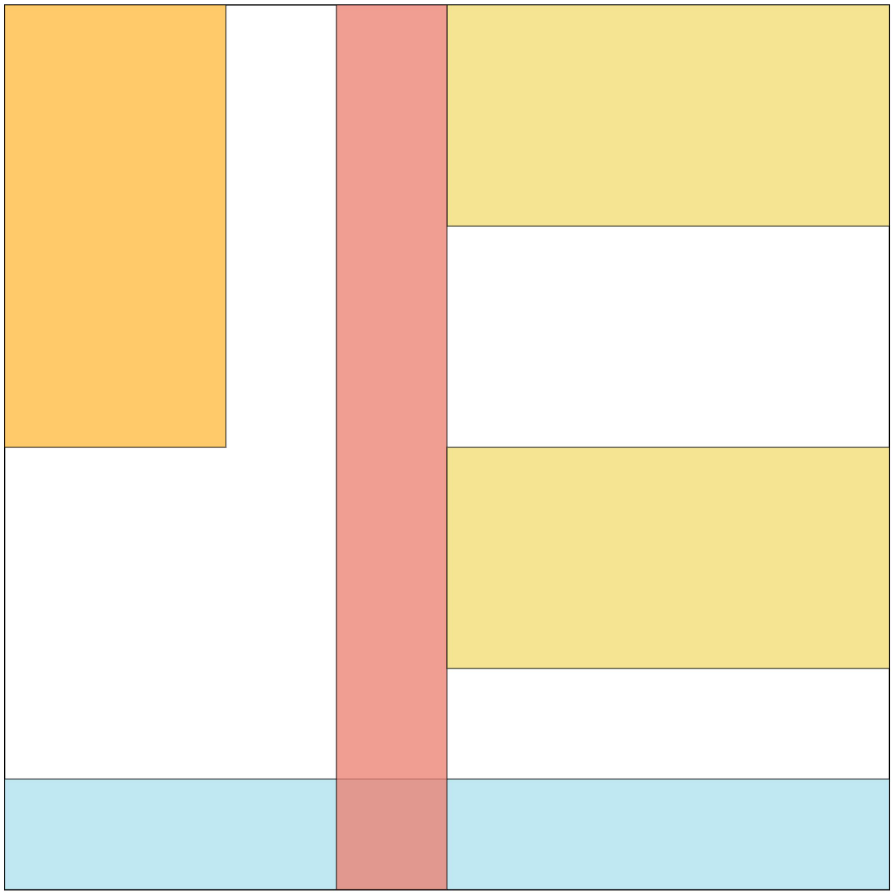
$$\|\mathcal{D}(\mathcal{F}_n; \cdot)\|_\infty \asymp n \asymp \log F_n.$$

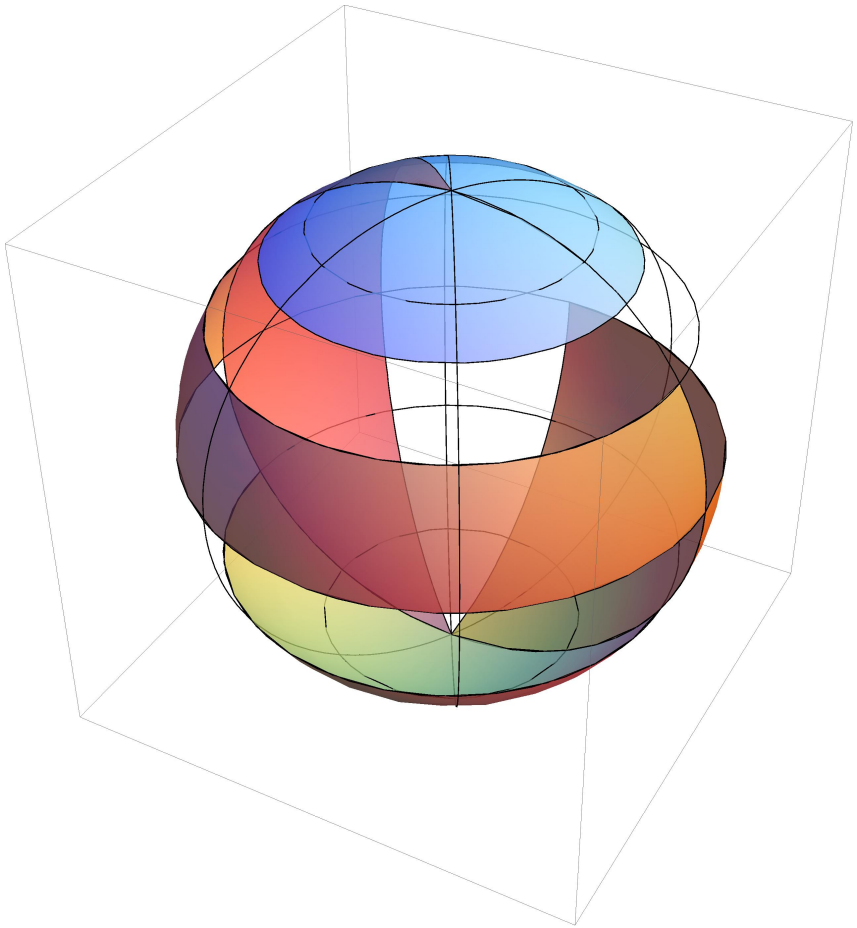
$\{x\}$ is fractional part of real x.



$n = 14$: $F_n = 377$.

Area preserving **Lambert transformation** $\Phi : [0,1]^2 \to \mathbb{S}^2$





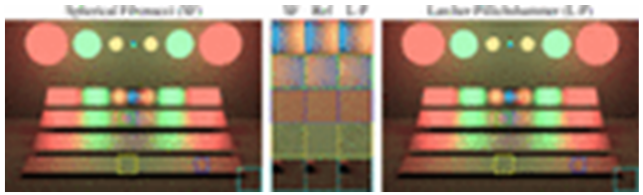
$$\Phi(x,y) = \begin{pmatrix} 2\cos(2\pi y)\sqrt{x-x^2} \\ 2\sin(2\pi y)\sqrt{x-x^2} \\ 1-2x \end{pmatrix}$$

Return

Spherical Fibonacci Point Sets for Illumination Integrals (pages 134–143)

R. Marques, C. Bouville, M. Ribardière, L. P. Santos and K. Bouatouch

Article first published online: 24 JUL 2013 | DOI: 10.1111/cgf.12190



Quasi-Monte Carlo (QMC) methods exhibit a faster convergence rate than that of classic Monte Carlo methods. This feature has made QMC prevalent in image synthesis, where it is frequently used for approximating the value of spherical integrals (e.g. illumination integral). The common approach for generating QMC sampling patterns for spherical integration is to resort to unit square low-discrepancy sequences and map them to the hemisphere. However such an approach is suboptimal as these sequences do not account for the spherical topology and their discrepancy properties on the unit square are impaired by the spherical projection.

Estimates for s^* for $d = 2$

Return

$$s^* := \sup \left\{ s : \begin{array}{l} (X_N) \text{ is QMC design} \\ \text{sequence for } \mathbb{H}^s(\mathbb{S}^d) \end{array} \right\}.$$

Table: Estimates of s^* for $d = 2$

Point set	s^*
Fekete	1.5
Equal area	2
Coulomb energy	2
Log energy	3
Generalized spiral	3
Distance	4
Spherical designs	∞